HOMEWORK 4

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14.5

30. $\nabla f = \frac{2y}{(x+y)^2}\mathbf{i} - \frac{2x}{(x+y)^2}\mathbf{j}$

(a) $\nabla f\left(-\frac{1}{2}, \frac{3}{2}\right) = 3\mathbf{i} + \mathbf{j} \Rightarrow \left|\nabla f\left(-\frac{1}{2}, \frac{3}{2}\right)\right| = \sqrt{10} \Rightarrow D_{\mathbf{u}}f\left(-\frac{1}{2}, \frac{3}{2}\right) = \sqrt{10}$ in the direction of $\mathbf{u} = \frac{3}{\sqrt{10}}\mathbf{i} + \frac{1}{\sqrt{10}}\mathbf{j}$

(b) $-\nabla f\left(-\frac{1}{2}, \frac{3}{2}\right) = -3\mathbf{i} - \mathbf{j} \Rightarrow \left|\nabla f\left(-\frac{1}{2}, \frac{3}{2}\right)\right| = \sqrt{10} \Rightarrow D_{\mathbf{u}}f(1, -1) = -\sqrt{10}$ in the direction of $\mathbf{u} = -\frac{3}{\sqrt{10}}\mathbf{i} - \frac{1}{\sqrt{10}}\mathbf{j}$

(c) $D_{\mathbf{u}}f\left(-\frac{1}{2}, \frac{3}{2}\right) = 0$ in the direction of $\mathbf{u} = \frac{1}{\sqrt{10}}\mathbf{i} - \frac{3}{\sqrt{10}}\mathbf{j}$ or $\mathbf{u} = -\frac{1}{\sqrt{10}}\mathbf{i} + \frac{3}{\sqrt{10}}\mathbf{j}$

(d) Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} \Rightarrow |\mathbf{u}| = \sqrt{u_1^2 + u_2^2} = 1 \Rightarrow u_1^2 + u_2^2 = 1$; $D_{\mathbf{u}}f\left(-\frac{1}{2}, \frac{3}{2}\right) = \nabla f\left(-\frac{1}{2}, \frac{3}{2}\right) \cdot \mathbf{u} = (3\mathbf{i} + \mathbf{j}) \cdot (u_1\mathbf{i} + u_2\mathbf{j})$
$= 3u_1 + u_2 = -2 \Rightarrow u_2 = -3u_1 - 2 \Rightarrow u_1^2 + (-3u_1 - 2)^2 = 1 \Rightarrow 10u_1^2 + 12u_1 + 3 = 0 \Rightarrow u_1 = \frac{-6 \pm \sqrt{6}}{10}$
$u_1 = \frac{-6+\sqrt{6}}{10} \Rightarrow u_2 = \frac{-2-3\sqrt{6}}{10} \Rightarrow \mathbf{u} = \frac{-6+\sqrt{6}}{10}\mathbf{i} + \frac{-2-3\sqrt{6}}{10}\mathbf{j}$, or $u_1 = \frac{-6-\sqrt{6}}{10} \Rightarrow u_2 = \frac{-2+3\sqrt{6}}{10}$
$\Rightarrow \mathbf{u} = \frac{-6-\sqrt{6}}{10}\mathbf{i} + \frac{-2+3\sqrt{6}}{10}\mathbf{j}$

(e) Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} \Rightarrow |\mathbf{u}| = \sqrt{u_1^2 + u_2^2} = 1 \Rightarrow u_1^2 + u_2^2 = 1$; $D_{\mathbf{u}}f\left(-\frac{1}{2}, \frac{3}{2}\right) = \nabla f\left(-\frac{1}{2}, \frac{3}{2}\right) \cdot \mathbf{u} = (3\mathbf{i} + \mathbf{j}) \cdot (u_1\mathbf{i} + u_2\mathbf{j})$
$= 3u_1 + u_2 = 1 \Rightarrow u_2 = 1 - 3u_1 \Rightarrow u_1^2 + (1 - 3u_1)^2 = 1 \Rightarrow 10u_1^2 - 6u_1 = 0 \Rightarrow u_1 = 0$ or $u_1 = \frac{3}{5}$;
$u_1 = 0 \Rightarrow u_2 = 1 \Rightarrow \mathbf{u} = \mathbf{j}$, or $u_1 = \frac{3}{5} \Rightarrow u_2 = -\frac{4}{5} \Rightarrow \mathbf{u} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$

14.8

33. Let $g_1(x, y, z) = 2x - y = 0$ and $g_2(x, y, z) = y + z = 0 \Rightarrow \nabla g_1 = 2\mathbf{i} - \mathbf{j}$, $\nabla g_2 = \mathbf{j} + \mathbf{k}$, and $\nabla f = 2x\mathbf{i} + 2\mathbf{j} - 2z\mathbf{k}$ so that $\nabla f = \lambda \nabla g_1 + \mu \nabla g_2 \Rightarrow 2x\mathbf{i} + 2\mathbf{j} - 2z\mathbf{k} = \lambda(2\mathbf{i} - \mathbf{j}) + \mu(\mathbf{j} + \mathbf{k}) \Rightarrow 2x\mathbf{i} + 2\mathbf{j} - 2z\mathbf{k} = 2\lambda\mathbf{i} + (\mu - \lambda)\mathbf{j} + \mu\mathbf{k}$
$\Rightarrow 2x = 2\lambda$, $2 = \mu - \lambda$, and $-2z = \mu \Rightarrow x = \lambda$. Then $2 = -2z - x \Rightarrow x = -2z - 2$ so that $2x - y = 0$
$\Rightarrow 2(-2z - 2) - y = 0 \Rightarrow -4z - 4 - y = 0$. This equation coupled with $y + z = 0$ implies $z = -\frac{4}{3}$ and $y = \frac{4}{3}$. Then $x = \frac{2}{3}$ so that $\left(\frac{2}{3}, \frac{4}{3}, -\frac{4}{3}\right)$ is the point that gives the maximum value $f\left(\frac{2}{3}, \frac{4}{3}, -\frac{4}{3}\right) = \left(\frac{2}{3}\right)^2 + 2\left(\frac{4}{3}\right) - \left(-\frac{4}{3}\right)^2 = \frac{4}{3}$.

37. Let $g_1(x, y, z) = z - 1 = 0$ and $g_2(x, y, z) = x^2 + y^2 + z^2 - 10 = 0 \Rightarrow \nabla g_1 = \mathbf{k}$, $\nabla g_2 = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$, and $\nabla f = 2xyz\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k}$ so that $\nabla f = \lambda \nabla g_1 + \mu \nabla g_2 \Rightarrow 2xyz\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k} = \lambda(\mathbf{k}) + \mu(2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k})$
$\Rightarrow 2xyz = 2x\mu$, $x^2z = 2y\mu$, and $x^2y = 2z\mu + \lambda \Rightarrow xyz = x\mu \Rightarrow x = 0$ or $yz = \mu \Rightarrow \mu = y$ since $z = 1$.
CASE 1: $x = 0$ and $z = 1 \Rightarrow y^2 - 9 = 0$ (from g_2) $\Rightarrow y = \pm 3$ yielding the points $(0, \pm 3, 1)$.
CASE 2: $\mu = y \Rightarrow x^2z = 2y^2 \Rightarrow x^2 = 2y^2$ (since $z = 1$) $\Rightarrow 2y^2 + y^2 + 1 - 10 = 0$ (from g_2) $\Rightarrow 3y^2 - 9 = 0$
$\Rightarrow y = \pm\sqrt{3} \Rightarrow x^2 = 2\left(\pm\sqrt{3}\right)^2 \Rightarrow x = \pm\sqrt{6}$ yielding the points $\left(\pm\sqrt{6}, \pm\sqrt{3}, 1\right)$.
Now $f(0, \pm 3, 1) = 1$ and $f\left(\pm\sqrt{6}, \pm\sqrt{3}, 1\right) = 6\left(\pm\sqrt{3}\right) + 1 = 1 \pm 6\sqrt{3}$. Therefore the maximum of f is $1 + 6\sqrt{3}$ at $\left(\pm\sqrt{6}, \sqrt{3}, 1\right)$, and the minimum of f is $1 - 6\sqrt{3}$ at $\left(\pm\sqrt{6}, -\sqrt{3}, 1\right)$.

43. (a) Maximize $f(a, b, c) = a^2b^2c^2$ subject to $a^2 + b^2 + c^2 = r^2$. Thus $\nabla f = 2ab^2c^2\mathbf{i} + 2a^2bc^2\mathbf{j} + 2a^2b^2c\mathbf{k}$ and $\nabla g = 2a\mathbf{i} + 2b\mathbf{j} + 2c\mathbf{k}$ so that $\nabla f = \lambda \nabla g \Rightarrow 2ab^2c^2 = 2a\lambda$, $2a^2bc^2 = 2b\lambda$, and $2a^2b^2c = 2c\lambda$
$\Rightarrow 2a^2b^2c^2 = 2a^2\lambda = 2b^2\lambda = 2c^2\lambda \Rightarrow \lambda = 0$ or $a^2 = b^2 = c^2$.
CASE 1: $\lambda = 0 \Rightarrow a^2b^2c^2 = 0$.
CASE 2: $a^2 = b^2 = c^2 \Rightarrow f(a, b, c) = a^2a^2a^2$ and $3a^2 = r^2 \Rightarrow f(a, b, c) = \left(\frac{r^2}{3}\right)^3$ is the maximum value.

(b) The point $\left(\sqrt{a}, \sqrt{b}, \sqrt{c}\right)$ is on the sphere if $a + b + c = r^2$. Moreover, by part (a), $abc = f\left(\sqrt{a}, \sqrt{b}, \sqrt{c}\right)$
$\le \left(\frac{r^2}{3}\right)^3 \Rightarrow (abc)^{1/3} \le \frac{r^2}{3} = \frac{a+b+c}{3}$, as claimed.

14.9

5. $f(x, y) = e^x \ln(1 + y) \Rightarrow f_x = e^x \ln(1 + y)$, $f_y = \frac{e^x}{1+y}$, $f_{xx} = e^x \ln(1 + y)$, $f_{xy} = \frac{e^x}{1+y}$, $f_{yy} = -\frac{e^x}{(1+y)^2}$
$\Rightarrow f(x, y) \approx f(0, 0) + xf_x(0, 0) + yf_y(0, 0) + \frac{1}{2}\left[x^2f_{xx}(0, 0) + 2xyf_{xy}(0, 0) + y^2f_{yy}(0, 0)\right]$
$= 0 + x \cdot 0 + y \cdot 1 + \frac{1}{2}\left[x^2 \cdot 0 + 2xy \cdot 1 + y^2 \cdot (-1)\right] = y + \frac{1}{2}\left(2xy - y^2\right)$, quadratic approximation;
$f_{xxx} = e^x \ln(1 + y)$, $f_{xxy} = \frac{e^x}{1+y}$, $f_{xyy} = -\frac{e^x}{(1+y)^2}$, $f_{yyy} = \frac{2e^x}{(1+y)^3}$

$$\Rightarrow f(x,y) \approx \text{quadratic} + \tfrac{1}{6}\left[x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right]$$
$$= y + \tfrac{1}{2}(2xy - y^2) + \tfrac{1}{6}\left[x^3 \cdot 0 + 3x^2 y \cdot 1 + 3xy^2 \cdot (-1) + y^3 \cdot 2\right]$$
$$= y + \tfrac{1}{2}(2xy - y^2) + \tfrac{1}{6}(3x^2 y - 3xy^2 + 2y^3), \text{ cubic approximation}$$

8. $f(x,y) = \cos(x^2+y^2) \Rightarrow f_x = -2x\sin(x^2+y^2),\ f_y = -2y\sin(x^2+y^2),$
$f_{xx} = -2\sin(x^2+y^2) - 4x^2\cos(x^2+y^2),\ f_{xy} = -4xy\cos(x^2+y^2),\ f_{yy} = -2\sin(x^2+y^2) - 4y^2\cos(x^2+y^2)$
$\Rightarrow f(x,y) \approx f(0,0) + xf_x(0,0) + yf_y(0,0) + \frac{1}{2}\left[x^2 f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$
$= 1 + x\cdot 0 + y \cdot 0 + \frac{1}{2}\left[x^2\cdot 0 + 2xy\cdot 0 + y^2 \cdot 0\right] = 1$, quadratic approximation;
$f_{xxx} = -12x\cos(x^2+y^2) + 8x^3\sin(x^2+y^2),\ f_{xxy} = -4y\cos(x^2+y^2) + 8x^2 y\sin(x^2+y^2),$
$f_{xyy} = -4x\cos(x^2+y^2) + 8xy^2\sin(x^2+y^2),\ f_{yyy} = -12y\cos(x^2+y^2) + 8y^3\sin(x^2+y^2)$
$\Rightarrow f(x,y) \approx \text{quadratic} + \frac{1}{6}\left[x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right]$
$= 1 + \frac{1}{6}(x^3\cdot 0 + 3x^2 y \cdot 0 + 3xy^2 \cdot 0 + y^3 \cdot 0) = 1$, cubic approximation

7 $f = e^{2x}\sin(3y)$

a. $e^{2x} = 1 + 2x + \frac{(2x)^2}{2!} + \frac{(3x)^3}{3!} + \cdots$

$\sin(3y) = 3y - \frac{(3y)^3}{3!}.$

$e^{2x}\sin(3y) = (1 + 2x + 2x^2 + \cdots)(3y - \frac{9}{2}y^3)$

$= 3y + 6xy + 6x^2 y - \frac{9}{2}y^3$

$P_3(x,y) = 3y + 6xy + 6x^2 y - \frac{9}{2}y^3$

b. $|f - P_m|(x,y) \le E_{m+1}(x,y)$

$E_n(x,y) = \frac{1}{n!}\max_{x^2+y^2<4}\{|\partial_x^n f|, \cdots, |\partial_y^n f|\}(x^2+y^2)^{\frac{n}{2}}$

$|\partial_y^{n-k}\partial_x^k f| \le 3^{n-k}2^k e^{2x} \le 3^{n-k}2^k e^4 \le 3^n e^4$

$k = 0, \cdots, n$

$E_n(x,y) < \frac{1}{n!}3^n e^4 4^{\frac{n}{2}} < \frac{1}{100}$

when $n \geq 22$

$E_n(x,y) < 0.007$

So $|f - P_{21}(x,y)| < 0.01$

for any (x,y) satisfying $x^2+y^2<4$